PROPERTIES OF CONNECTIVITY MAPPINGS AND CERTAIN OTHER NONCONTINUOUS FUNCTIONS

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PREFACE

This paper will be primarily concerned with a particular collection of noncontinuous mappings which includes, as special cases, the connected mappings, connectivity maps, and peripherally continuous transformations. In addition, the sets of points of discontinuity of the connectivity maps are investigated. Chapter I is an introductory chapter giving the definitions of the above mentioned functions and of certain other notions pertinent to this study. In Chapter II two collections of noncontinuous mappings are defined and the relationship between them considered. Chapter III extends the considerations of Chapter II to the graph mappings and to certain other mappings induced by these noncontinuous transformations. The material of Chapter IV is devoted to the study of the discontinuities of connectivity maps and peripherally continuous transformations. Chapter V contains a summary of the results.

I should like to express my sincere appreciation to all members of my advisory committee, and especially to Professor O. H. Hamilton for his wise counsel and patient guidance in the preparation of this paper; to Dr. L. Wayne Johnson for my assistantships and many other considerations; and, most of all to my wife, Maxine.

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CHAPTER I

INTRODUCTION

This paper will be devoted to the development of certain properties of connectivity mappings, peripherally continuous mappings, a class of mappings which includes the connectivity and peripherally continuous mappings, and the relationships among these noncontinuous mappings, their induced graph mappings and other mappings associated with their graphs. In particular, some properties of the sets of points of discontinuity of connectivity mappings will be investigated. The concept of the connectivity mapping was defined by John Nash in [7] and the peripherally continuous mapping was defined by O. H. Hamilton in [4]. In [8] John Stallings extended a result of Hamilton and proved that, under certain conditions on the topological spaces involved, every peripherally continuous mapping is a connectivity mapping. Hagan [2] proved the converse of this result and showed in particular that, on topological n-cells of dimension greater than one, connectivity mappings and peripherally continuous mappings are the same. In [6] Long developed certain properties and relationships among these and other mappings and suggested that a larger class of noncontinuous mappings would retain many of the properties of connectivity and peripherally continuous mappings. It was this suggestion and the papers [2] and [4] that led to the investigations contained in this paper.

In Chapter II the F_1-mappings and F_2-mappings are defined and their relationships to continuous mappings and connected mappings are investigated. A property of peripherally continuous mappings given by Long [6, p. 5] is shown to characterize the F_2-mappings and sufficient conditions that a mapping be an F_1-mapping are given. It is shown that the limit of a uniformly convergent sequence of F_2-mappings is an F_2-mapping. The latter part of the chapter is concerned with the composites of these mappings and one result is that the class of F_1-mappings is closed under the operation of composition but the class of F_2-mappings is not closed under this operation.

In Chapter III the projection mapping h induced by a mapping f is defined and attention is focused on the connections between these functions. It is shown that on a suitably restricted family of spaces these mappings are F_2-mappings and that continuity of f is implied by the peripheral continuity of h. Some conditions are given that imply the continuity of a peripherally continuous projection mapping induced by a mapping f. In addition conditions under which one-to-one, and peripherally continuous mappings will be continuous are given.

Chapter IV is concerned primarily with the set of points of discontinuity of connectivity mappings and with the graphs of these mappings. It is demonstrated that on certain spaces S into the real numbers R either the set of points of discontinuity of a connectivity mapping f is a set of the first category or the graph of f is dense in some open subset $S \times R$. A result showing the existence of everywhere-discontinuous connectivity mappings on certain spaces S into the reals is one of the main results of this chapter.

The definition of a connected mapping and those of the connectivity and peripherally continuous mappings will now be given.

<u>Definition 1.1.</u> A mapping f of a topological space S into a topological space T is said to be connected if for every connected subset M of S it is true that f(M) is connected.

<u>Definition 1.2.</u> Let f be a mapping of a set S into a set T and let g be the mapping of S into $S \times T$ defined by $g(p) = (p, f(p))$. Then g is called the graph mapping induced by f.

<u>Definition 1.3.</u> Let f be a mapping of a topological space S into a topological space T and let g be the graph mapping induced by f. Then f is said to be a connectivity mapping if and only if g is a connected mapping.

<u>Definition 1.4.</u> A mapping f of a topological space S into a topological space T is called peripherally continuous if and only if for each point p in S and each pair of open sets U and V containing p and f(p), respectively, there is an open set D contained in U and containing p such that f maps the boundary of D into V.

<u>Definition 1.5.</u> A mapping f from a topological space S into a topological space T is called a local connectivity map if there is a covering $\{U_x\}$ of S by open sets such that f restricted to U_x is a connectivity mapping for every x.

If U is a subset of a space S the boundary of U will be denoted by F(U). Other definitions will be given as needed throughout this paper.

CHAPTER II

PROPERTIES OF CERTAIN DISCONTINUOUS MAPPINGS

If f is a mapping of the topological space S into the topological space T then f is continuous on S if and only if for every closed subset C of S, $f^{-1}(C)$ is closed in T. In [4] Hamilton has shown that if f is either a connectivity or a peripherally continuous mapping of S into T and C is a closed subset of T then $f^{-1}(C)$ has closed components in S. It is shown in what follows that if f is a connected mapping of S into T and if C is a closed subset of T then $f^{-1}(C)$ has closed components in S.

In this chapter an investigation of two classes of discontinuous mappings will be made. The second of these is defined by the property described above and includes the peripherally continuous, connectivity, and connected mappings.

Definition 2.1. A mapping f of a topological space S into a topological space T is called an F_1-mapping if and only if for every subset C of T having closed components in T it is true that $f^{-1}(C)$ has closed components in S.

Definition 2.2. A mapping f of a topological space S into a topological space T is called an F_2-mapping if and only if for every closed subset C of T it is true that $f^{-1}(C)$ has closed components in S.

The following sequence of theorems establishes set inclusion relations among the classes of continuous mappings, connected mappings, F_1-mappings, and F_2-mappings defined on a topological space S and having values in a T_1 space T.

Theorem 2.1. If f is a connected mapping of a topological space S into a T_1 space T then f is an F_1-mapping of S into T.

Proof. Let C be a subset of T having closed components and let M be a component of $f^{-1}(C)$. Let p be any limit point of M. Then $M \cup \{p\}$ is connected. Since f is a connected mapping, f(M) and $f(M \cup \{p\}) = f(M) \cup \{f(p)\}$ are connected. Therefore f(M) is a subset of some component K of C. Since, by hypothesis, K is closed and f is connected, it follows that f(p) is in K. For otherwise the closure of the connected set $f(M) \cup \{f(p)\}$ would be the union of $\overline{f(M)}$ and $\{f(p)\}$, which are disjoint closed sets since T is a T_1 space. But this would contradict the connectedness of f. Thus, since f(p) is in K, a component of C, p is in $f^{-1}(C)$. But since p is a limit point of the component M of $f^{-1}(C)$, then p is in M. Consequently M is closed and therefore f is an F_1-mapping.

Theorem 2.2. If f is an F_1-mapping of the topological space S into the topological space T then f is an F_2-mapping of S into T.

Proof. Let f be an F_1-mapping of S into T and C a closed subset of T. Then C has closed components in T and therefore $f^{-1}(C)$ has closed components in S. Hence f is an F_2-mapping of S into T.

Theorem 2.3. Let F_o, M, F_1 and F_2 denote the collection of continuous mappings, connected mappings, F_1-mappings and F_2-mappings of a topological space S into a T_1 space T. Then $F_o \subset M \subset F_1 \subset F_2$.

Proof. The proof is a restatement of Theorems 2.1, 2.2 and the well-known fact that a continuous mapping is a connected mapping.

The following two theorems provide necessary and sufficient conditions that a mapping of a topological space S into a topological space T be an F_2-mapping and sufficient conditions that a mapping of a topological space S into a topological space T be an F_1-mapping.

Theorem 2.4. A mapping f of a topological space S into a topological space T is an F_2-mapping if and only if for every connected subset M of S it is true that $f(\overline{M}) \subset \overline{f(M)}$.

Proof. Suppose that f is an F_2-mapping of S into T and that M is a connected subset of S. Let p be any limit point of M not in M. Assume that f(p) is not in $\overline{f(M)}$. Since f is an F_2-mapping $f^{-1}(\overline{f(M)})$ has closed components in S. Since M is connected then M is a subset of one of these components. Denote it by K. Then p must also belong to K since p is a limit point of M. Consequently f(p) belongs to $\overline{f(M)}$. Thus the assumption that f(p) is not in $\overline{f(M)}$ leads to the contradiction that f(p) is in $\overline{f(M)}$. It follows that, if p is a limit point of M not in M, then f(p) is in $\overline{f(M)}$ and, therefore that $f(\overline{M}) \subset \overline{f(M)}$.

Suppose next that f is a mapping of S into T satisfying the condition that $f(\overline{M}) \subset \overline{f(M)}$, for every connected subset M of S. Let C be a closed subset of T and let M be a component of $f^{-1}(C)$. Since $f(\overline{M}) \subset \overline{f(M)} \subset \overline{C}$ and C is closed then $\overline{M} \subset f^{-1}(C)$. Since M is a connected subset of $f^{-1}(C)$ and M is a component of $f^{-1}(C)$ then $M = \overline{M}$. This proves that f is an F_2-mapping.

Theorem 2.5. Let f be a mapping of the topological space S into the topological space T. In order that f be an F_1-mapping of S into T it is sufficient that, for every connected subset M of S, f(M) is a subset of the union of the closures of the components of f(M).

Proof. Let f be a mapping of S into T satisfying the hypothesis of the theorem and let C be a subset of T having closed components in

T. Let M be a component of $f^{-1}(C)$. Let G be the collection of components of f(M) and let F be the collection of components of C. If p is any limit point of M then, by hypothesis, f(p) belongs to the closure of some element K of G. Since K is a connected subset of C then K is a subset of some element H of F. By hypothesis, the elements of F are closed. Hence f(p) is in C and therefore p is in $f^{-1}(C)$. Therefore, since $M \cup \{p\}$ is a connected subset of $f^{-1}(C)$ and M is a component of $f^{-1}(C)$, then p belongs to M. Hence M is closed and consequently f is an F_1-mapping of S into T.

The mapping f of the closed interval $[0,1]$ of real numbers into the real numbers defined by $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$ is a connected mapping which is not continuous. The following examples show that the remaining set inclusions of Theorem 2.3 are also proper inclusions.

Example 2.1. Let f be the mapping of $[0,1]$ into itself defined by $f(x) = 1/4$ if x is rational and $f(x) = 3/4$ if x is irrational. Then f is not a connected mapping. However f is an F_1-mapping. For let C be any subset of $[0,1]$ having closed components. Let $A = \{1/4, 3/4\}$. If $C \cap A = \emptyset$ then $f^{-1}(C) = \emptyset$. If $C \cap A = \{1/4\}$ then $f^{-1}(C)$ is the set of rational numbers in $[0,1]$. If $C \cap A = \{3/4\}$ then $f^{-1}(C)$ is the set of irrational numbers in $[0,1]$. If $C \cap A = A$ then $f^{-1}(C) = [0,1]$. Thus, in any case $f^{-1}(C)$ has closed components.

The following example shows that an F_2-mapping of a topological space S into a topological space T may fail to be an F_1-mapping.

Example 2.2. Let S be the subset of the Euclidean plane E_2 where $0 \leq x \leq 1$ and $y = 0$ together with those points (x,y) of E_2 satisfying $0 \leq x \leq 1$, $0 \leq y \leq 1$ and x rational. Let T be the subset of S obtained by deleting those points $(x,0)$ of S where x is irrational. Let f be the mapping of S into T defined by $f(x,y) = (1,0)$ if x is irrational and $f(x,y) = (x,y)$ otherwise. The topologies for S and T are those induced by the usual topology for the plane. To show that f is an F_2-mapping let M be any connected subset of S and (x,y) any limit point of M. If $y > 0$ then $f(x,y)$ is also a limit point of $f(M)$ since in this case (x,y) is a limit point of the subset M^* of M consisting of all (x,y) in M with $y > 0$, since $f(M^*) = M^*$, and since $f(x,y) = (x,y)$. Suppose next that $y = 0$. If x is rational and $(x,0)$ is a limit point of M^* then $f(x,0) = (x,0)$ is again a limit point of $f(M^*) = M^*$. If x is rational and $(x,0)$ is not a limit point of M^* then M must contain points of the form $(z,0)$ for values of z arbitrarily close to x. Since $(x,0)$ is not a limit point of M^* there exists a disk D with center $(x,0)$ and radius r less than either of x or $1 - x$ such that $D \cap M^* = \emptyset$. It follows that M contains an interval I of points of the form $(z,0)$ with $(x,0)$ as an end point. Otherwise there would exist two numbers z_1 and z_2 such that $x - r < z_1 < x < z_2 < x+r$ if $0 < x < 1$, one number z_1 such that $x - r < z_1 < x$ if $x = 1$, or one number z_2 such that $x < z_2 < x + r$ if $x = 0$. Whatever the case M would be seperated in S by the points of S on the line segments joining $(z_1,0)$ to (x,r) and (x,r) to $(z_2,0)$, by the points of S on the line segment joining $(z_1,0)$ to (x,r), or by the points of S on the line segment joining (x,r) to $(z_2,0)$. The point $f(x,0) = (x,0)$ is then a limit point of the set of images under f of those points $(z,0)$ where z is rational. Hence $f(x,0)$ is a limit point of $f(M)$. Suppose now that $(x,0)$ is a limit point of M not in M and that x is irrational. If M contains a point $(x^1,0)$ with x^1 irrational then

$f(x,0) = f(x^1,0) = (1,0)$ is in $f(M)$. Suppose that M contains no point $(x^1,0)$ with x^1 irrational. Since $(x,0)$ is a limit point of M and S contains no points (x,y) with $y \neq 0$, then there exists two points (x_1,y_1) and (x_2,y_2) of M such that $x_1 \neq x_2$. Let x_3 be an irrational number between x_1 and x_2. Then $S - \{(x_3,0)\}$ is the union of two separated sets whose union contain M and each of which contain points of M. This contradicts the fact that M is connected. Hence if M has a limit point $(x,0)$ and x is irrational then $f(x,0)$ is in $\overline{f(M)}$. This completes the proof that if (x,y) is a limit point of M then $f(x,y)$ is in $\overline{f(M)}$. Hence $f(\overline{M}) \subset \overline{f(M)}$ and therefore f is an F_2-mapping, by Theorem 2.4.

To show that f is not an F_1-mapping consider the subset C of T consisting of those points (x,y) such that $x < 1/2$ together with the point $(1,0)$. Let J_x denote the set of all points in T with abscissa x. Then a set K is a component of C if and only if $K = J_x$ for some $x < 1/2$ or if $K = \{(1,0)\}$. Each of the components of C is closed in T. However not all of the components of $f^{-1}(C)$ are closed in S. Let M be the subset of S consisting of all points (x,y) in S such that $x < 1/2$. If (x,y) is in M and x is rational then $f(x,y)$ is in C. If (x,y) is in M and x is irrational then $y = 0$ and $f(x,y) = (1,0)$ is also in C. Now M is a connected subset of $f^{-1}(C)$ since M is the union of the collection of connected sets J_x, $0 \leq x < 1/2$, and the connected set J consisting of the points $(x,0)$ where $0 \leq x < 1/2$ and since each of sets J_x contains a point of J. To show that M is a component of $f^{-1}(C)$ let (x,y) be any point of S not in M such that $f(x,y)$ is in C. Then $x > 1/2$ since $f(1/2,y) = (1/2,y)$ is not in C for any y. Hence M is separated from (x,y) in $f^{-1}(C)$ by $J_{1/2}$. Therefore M is a component of $f^{-1}(C)$. M is not closed in S since (1/2,0) is a limit point of M not in M. This completes the proof that f is not an F_1-mapping of S into T.

If f is a continuous mapping of a topological space S into a topological space T and f is also a closed mapping then $f(\overline{M}) = \overline{f(M)}$ for every subset M of S. If f is a closed F_2-mapping of S into T a similar property is satisfied by f as is demonstrated by the following theorem.

Theorem 2.6. Let f be a closed F_2-mapping of the topological space S into the topological space T. Then for every connected subset M of S it is true that $f(\overline{M}) = \overline{f(M)}$.

Proof. Let M be a connected subset of S. The relation $f(\overline{M}) \subset \overline{f(M)}$ follows from Theorem 2.4. Hence the relation $f(M) \subset f(\overline{M}) \subset \overline{f(M)}$ together with the fact that f(M) is closed implies that $f(\overline{M}) = \overline{f(M)}$.

Theorem 2.7. Let $\{f_n\}$ be a sequence of F_2-mappings of the topological space S into the metric space T and suppose that $\{f_n\}$ converges uniformly to the mapping f of S into T. Then f is an F_2-mapping.

Proof. Let C be a closed subset of T, M a component of $f^{-1}(C)$, and p a limit point of M. Let ϵ be any positive number. Since $\lim f_n(p) = f(p)$ there exists an integer N_1 such that if $n > N_1$ then $d[f(p),f_n(p)] < \epsilon/3$, where d is the metric for T. Since $\{f_n\}$ converges uniformly to f on S there exists an integer N_2 such that $n > N_2$ implies that $d[f_n(x),f(x)] < \epsilon/3$, for every x in S. Since each of the mappings f_n is an F_2-mapping then $f_n(p)$ is in $\overline{f_n(M)}$. Let N be an integer larger than N_1N_2. Then there exists a point x_o such that

$d[f_N(p), f_N(x_o)] < \epsilon/3.$

Hence

$$d[f(p), f(x_o)] \leq d[f(p), f_N(p)] + d[f_N(p), f(x_o)]$$

$$\leq d[f(p), f_N(p)] + d[f_N(p), f_N(x_o)] + d[f_N(x_o), f(x_o)]$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3.$$

That is, given $\epsilon > 0$, there exists a point x_o in M such that $d[f(p), f(x_o)] < \epsilon$. This implies that f(p) is in $\overline{f(M)}$. Therefore $f(\overline{M}) \subset \overline{f(M)}$ and f is an F_2-mapping of S into T.

The point-wise limit of a sequence of F_2-mappings is not necessarily an F_2-mapping as is demonstrated by the following example.

Example 2.3. Let $S = T = [0,1]$ and let $f_n(x) = x^n$ for each x in S. Each f_n is continuous and is therefore an F_2-mapping by Theorem 2.3. However $f = \lim f_n$ is not an F_2-mapping since, if $M = [0,1)$, then $f(\overline{M}) = f([0,1]) = \{0,1\}$ is not a subset of $\overline{f(M)} = \{0\}$, which is in disagreement with the conclusion of Theorem 2.4.

If f and g are continuous mappings or connected mappings then gf is also continuous or connected. The following theorems show that the composite of F_1-mappings is again of the same type and that the composite gf is an F_2-mapping if f is an F_1-mapping and g is an F_2-mapping.

Theorem 2.8. Let S, T, and U be topological spaces, f an F_1-mapping of S into T, and g an F_1-mapping of T into U. Then gf is an F_1-mapping of S into U.

Proof. Let C be a subset of U having closed components in U. Then $g^{-1}(C)$ has closed components in T and therefore $f^{-1}(g^{-1}(C))$ has closed components in S. The result that gf is an F_1-mapping of S into U follows from the equation $(gf)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Theorem 2.9. Let S, T, and U be topological spaces. Suppose that f is an F_1-mapping of S into T and that g is an F_2-mapping of T into U. Then the composite gf is an F_2-mapping of S into U.

Proof. If C is a closed subset of U then $g^{-1}(C)$ has closed components in T and since f is an F_1-mapping then $f^{-1}(g^{-1}(C)) = (gf)^{-1}(C)$ has closed components in S. Hence gf is an F_2-mapping.

The composite of two F_2-mappings is not necessarily an F_2-mapping and the composite gf of an F_1-mapping g and an F_2-mapping f may not be an F_2-mapping. This is shown by the following example.

Example 2.4. Let S and T be the spaces described in Example 2.2 and let f be the mapping S into T described in that example. Let U be the subspace of T consisting of all points (x,y) of T such that x = 0 or x = 1. Then $U = J_o \cup J_1$, where the meaning of J_x is the same as in Example 2.2. Let g be the mapping of T into U such that $g(x,y) = (1,y)$ if $0 \leq x < 1/2$ or if x = 1 and $g(x,y) = (0,y)$ otherwise. Then g is an F_1-mapping of T into U. For let C be a subset of U having closed components in U. Let M be any connected subset of $g^{-1}(C)$ and let (x,y) be a limit point of M. By the definitions of T, M is a sub-interval of one of the sets J_x with endpoints (x,y_1) and (x,y_2) and may contain one or both of its end points. Consequently, by the definition of g, $g(\overline{M})$ is the collection of all points (1,y) or the collection of all points (0,y), where $y_1 \leq y \leq y_2$. Hence $g(\overline{M})$ is a subset of the component of C that contains g(M) since, by hypothesis, that component of C is closed. It follows from Theorem 2.5 that g is an F_1-mapping.

To show that gf is not an F_2-mapping let M be the connected subset of S consisting of those points (x,y) of S such that $0 \leq x < 1/2$. The point (1/2,1/2) is a limit point of M but (gf)(1/2,1/2) = (0,1/2) is not a point of $\overline{(gf)(M)} = J_1$. Hence $(gf)(\overline{M})$ is not a subset of $\overline{(gf)(M)}$ and therefore, by Theorem 2.4, gf is not an F_2-mapping of S into U.

This example shows that the composite gf of an F_2-mapping f and an F_1-mapping g may not be an F_2-mapping. Since an F_1-mapping is also an F_2-mapping it also demonstrates that the composite of two F_2-mappings may not be an F_2-mapping.

CHAPTER III

PROPERTIES OF THE GRAPH MAPPINGS AND PROJECTION MAPPINGS INDUCED BY NONCONTINUOUS MAPPINGS

It is well known that if f is a mapping of the topological space S into the topological space T and g is the induced graph mapping of S into $S \times T$ then f is continuous if and only if g is a homeomorphism [3]. In [2] Hagan has shown that f is a peripherally continuous mapping of S into T if and only if g is a peripherally continuous mapping of S into $S \times T$. In this chapter it is shown that the analogous result holds for F_2-mappings defined on a restricted family of spaces S.

If f is a mapping of the topological space S into the topological space T and g is the induced graph mapping then, in general, there are many mappings h of $S \times T$ into T such that the composite hg is f. One of the main results of this chapter is that, for certain topological spaces S and T, and for a particular mapping h of $S \times T$ into T, which satisfies f = hg, either all of f, g, and h are F_2-mappings or none of them are F_2-mappings.

Theorem 3.1. Let f be a mapping of the topological space S into the topological space T and let g be the induced graph mapping of S into $S \times T$. Then f is an F_2-mapping if g is an F_2-mapping.

Proof. Let M be a connected subset of S and p any limit point of M not in M. Since g is an F_2-mapping, $g(p) = (p,f(p))$ is a limit point of $g(M)$. Since the projection mapping π of $S \times T$ onto T is continuous then $\pi(g(p)) = f(p)$ is a point or a limit point of $\pi(g(M)) = f(M)$. It follows from Theorem 2.4 that f is an F_2-mapping.

Definition 3.1. A topological space S is said to be hereditarily locally connected provided that every connected subset of S is locally connected.

Theorem 3.2. Let f be an F_2-mapping of the hereditarily locally connected space S into the topological space T. Then the induced graph mapping g of S into $S \times T$ is an F_2-mapping.

Proof. Let M be any connected subset of S and p any limit point of M not in M. Assume that $g(p)$ is not in $\overline{g(M)}$. Then there exists an open set U containing p and an open set V containing $f(p)$ such that $(U \times V) \cap (g(M)) = \emptyset$. Since S is hereditarily locally connected and $M \cup \{p\}$ is connected then $M \cup \{p\}$ is locally connected at p. Hence there exists an open set W containing p such that $N = W \cap (M \cup \{p\})$ is connected. Since f is an F_2-mapping and N is connected it follows that $f(\overline{N}) \subset \overline{f(N)}$. Since p is a limit point of M then p is also a limit point of N. Therefore V contains a point of $f(N - \{p\})$. If q is any point of $N - \{p\}$ such that $f(q)$ is in V then q belongs to U and therefore $g(q) = (q,f(q))$ is a point of $g(M)$ in $U \times V$. But this contradicts the equation $(U \times V) \cap g(M) = \emptyset$ derived from the assumption that $g(p)$ is not in $\overline{g(M)}$. Hence the assumption cannot hold and consequently $g(\overline{M}) \subset \overline{g(M)}$. Hence g is an F_2-mapping of S into $S \times T$.

Definition 3.2. Let S and T be topological spaces and let f be a mapping of S into T. The mapping h of $S \times T$ into T defined by

$h(x,y) = f(x)$ is called the projection mapping of $S \times T$ into T induced by f. In the case that only one such mapping f is under consideration the mapping h will be referred to as the induced projection mapping of $S \times T$ into T.

Theorem 3.3. Let f be a mapping of the topological space S into the topological space T and let h be the induced projection mapping of $S \times T$ into T. Then f is an F_2-mapping if and only if h is an F_2-mapping.

Proof. Assume first that f is an F_2-mapping of S into T and let M be a connected subset of $S \times T$. Let (p,q) be a limit point of M and let π denote the projection mapping of $S \times T$ onto S. Since π is continuous then $\pi(M)$ is connected and $\pi(p,q) = p$ is a point or a limit point of $\pi(M)$. Since f is an F_2-mapping then $f(p)$ is in $\overline{f(\pi(M))}$. For each (x,y) in $S \times T$, $h(x,y) = f(x) = f(\pi(x,y))$. Hence $h(p,q) = f(p)$ and $\overline{h(M)} = \overline{f(\pi(M))}$, so $h(p,q)$ is in $\overline{h(M)}$. Thus $h(\overline{M}) \subset \overline{h(M)}$ and therefore h is an F_2-mapping of $S \times T$ into T.

Suppose now that h is an F_2-mapping of $S \times T$ into T and let M be a connected subset of S. Let q be a point of T and let $Mq = M \times \{q\}$. Suppose that p is a limit point of M. Then (p,q) is a limit point of Mq and, furthermore, Mq is a connected subset of $S \times T$. Since h is an F_2-mapping, $h(p,q)$ is in $\overline{h(Mq)}$. But $h(p,q) = f(p)$ and $h(Mq) = f(M)$. Therefore $f(p)$ is in $\overline{f(M)}$ and consequently $f(\overline{M}) \subset \overline{f(M)}$. This proves that f is an F_2-mapping.

Theorem 3.4. Let f be a mapping of the hereditarily locally connected space S into the topological space T and let g and h be the induced graph mapping and induced projection mapping of S into $S \times T$ and of $S \times T$ into T, respectively. Then any particular one of these mappings is an F_2-mapping if and only if all of them are F_2-mappings.

<u>Proof.</u> If any one of the mappings f, g, or h is an F_2-mapping then Theorems 3.1, 3.2 and 3.3 imply that the other two are also F_2-mappings. Similarly, if any one of these mappings fails to be an F_2-mapping then the same theorems imply that the other two also fail to be F_2-mappings.

<u>Corollary 3.1.</u> If f is a connected, peripherally continuous, or connectivity mapping of the hereditarily locally connected space S into the topological space T then the induced graph mapping g and the induced projection mapping h are F_2-mappings.

<u>Proof.</u> If f is a connected, peripherally continuous, or connectivity mapping of S into T then f is an F_2-mapping of S into T.

<u>Theorem 3.5.</u> Let f be a mapping of the topological space S into the topological space T and let h be the induced projection mapping of $S \times T$ into T. Let (p,q) be a point in $S \times T$. Then h is continuous at (p,q) if and only if f is continuous at p.

<u>Proof.</u> Suppose first that f is continuous at the point p in S and let V be an open set containing $h(p,q) = f(p)$. Since f is continuous at p there exists an open set U containing p such that $f(U) \subset V$. Hence, for every point (x,y) such that x is in U, $h(x,y) = f(x)$ is in V. Therefore $U \times T$ is an open set containing (p,q) and $h(U \times T) = f(U) \subset V$. This proves that h is continuous at (p,q).

Suppose next that h is continuous at the point (p,q) in $S \times T$ and that V is an open set containing f(p). Then there exists an open set U in $S \times T$ containing (p,q) such that $h(U) \subset V$. Let H and K be open sets in S and T containing p and q such that $H \times K \subset U$. Let x be any point in H. Then, since q is in K, $f(x) = h(x,q)$ is in

$h(H \times K) \subset h(U)$. Therefore H is an open set containing p such that $f(H) \subset h(U) \subset V$ and consequently f is continuous at p.

If f is a mapping of a topological space S into a topological space T and g is the induced graph mapping then in order that g be a homeomorphism it is sufficient that g be continuous [3, p. 76]. The following theorem demonstrates that in order for the induced projection mapping h to be continuous it is sufficient that h be peripherally continuous and T connected.

Theorem 3.6. Let f be a mapping of the topological space S into the topological space T and let h be the induced projection mapping of $S \times T$ into T. If T is connected and h is peripherally continuous on $S \times T$ then h is continuous on $S \times T$.

Proof. Assume that h is not continuous at some point (p,q) in $S \times T$. Then it follows from Theorem 3.5 that f is not continuous at p. It will be shown that this implies that h is not peripherally continuous at (p,f(p)), which contradicts the hypothesis of the theorem.

Since f is not continuous at p there exists an open set V containing f(p) such that for every open set W containing p there is a point x in W such that f(x) is not in V. The set $S \times V$ is an open subset of $S \times T$ containing (p,f(p)). Let D be any open subset of $S \times V$ containing (p,f(p)) and let π denote the projection mapping of $S \times T$ onto S. Since π is an open mapping [5, p. 89] then $\pi(D)$ is an open set in S containing p. Therefore there exists a point x_o in $\pi(D)$ such that $f(x_o)$ is not in V.

Let Y be the collection of all points y of T such that (x_o,y) is in D. Then Y is an open subset of T. For let y be any point of Y. By definition of Y, the point (x_o,y) is in D. Since D is open in $S \times T$

there exists an open subset G of S containing x_o and an open subset H of T containing y such that $G \times H \subset D$. Hence H is an open set containing y such that, for each t in H, the point (x_o,t) is in D. Therefore H is an open subset of Y containing y. Since y was an arbitrary point of Y it follows that Y is open.

Consider any point y in Y. Then, since the point (x_o,y) is in D and since $D \subset S \times V$, it follows that Y is a subset of V. Now $T - V$ contains the point $f(x_o)$ and therefore $T - Y$ contains $f(x_o)$. Since T is connected, not both Y and $T - Y$ are open. Hence $T - Y$ contains a limit point y_1 of Y. It will be shown next that the point (x_o,y_1) is in the boundary $F(D)$ of D.

Let N be any open set in $S \times T$ containing (x_o,y_1). Then there exists an open subset K of S containing x_o and an open subset L of T containing y_1 such that $K \times L \subset N$. Since y_1 is a limit point of Y not in Y then L contains a point t of Y. Hence $K \times L$, and therefore N, contains the point (x_o,t) which belongs to D by definition of Y. Also, since y_1 is not in Y, it follows that $(x_o y_1)$ is not a point of D. Hence any open set N containing the point (x_o,y_1) of $(S \times T) - D$ contains a point (x_o,t) not in D. This proves that (x_o,y_1) is in $F(D)$. But $h(x_o,y_1) = f(x_o)$ is not in V.

Hence it has been shown that there exists an open set V containing $f(p) = (p,f(p))$ and an open set $S \times V$ containing $(p,f(p))$ such that if D is any open subset of $S \times V$ containing $(p,f(p))$ then $F(D)$ contains a point (x_o,y_1) such that $h(x_o,y_1)$ is not in V. This implies that h is not peripherally continuous at $(p,f(p))$, which contradicts the hypothesis of the theorem. Therefore h is continuous on $S \times T$.

The proof of the following corollary is an immediate consequence of Theorem 3.6 and [3, p. 76].

Corollary 3.2. Let f be a mapping of the topological space S into the topological space T and let g and h be the induced graph mapping and the induced projective mapping, respectively. Assume that T is connected. Then the following three statements are equivalent.

(1) h is a peripherally continuous mapping.

(2) f is a continuous mapping.

(3) g is a homeomorphism.

Definition 3.3 A space S is said to be locally peripherally connected at the point p if for every open set U containing p there is an open set V containing p and contained in U such that F(V) is connected. A space is locally peripherally connected if it is locally peripherally connected at each of its points.

In [8] Stallings proved the following generalization of a theorem of Hamilton [4].

Proposition 3.1. If f is a local connectivity mapping of the locally peripherally connected polyhedron P into a regular Hausdorff space Y, then f is peripherally continuous.

Theorem 3.7. Let f be a mapping of a topological space S into a connected regular Hausdorff space T and let h be the induced projection mapping of S x T into T. If h is a local connectivity mapping and S x T is a locally peripherally connected polyhedron, then h is continuous.

Proof. Proposition 3.1 implies that h is peripherally continuous and since T is connected it follows from Theorem 3.6 that h is continuous.

In [2, p. 7] Hagan gave an example that demonstrates that an open peripherally continuous mapping need not be continuous and in [6, p. 24] Long gave an example which shows that a one-to-one peripherally continuous mapping may fail to be continuous. The following theorem shows that if a peripherally continuous mapping f of S onto T is both open and one-to-one then f will be continuous provided that certain conditions are imposed on S and T.

Theorem 3.8. If f is a one-to-one, open and peripherally continuous mapping of the first countable T_1 space S onto the compact T_1 space T, then f is continuous.

Proof. Suppose that f is not continuous at some point p in S. Then there exists an open set V in T containing f(p) such that for every open set U in S containing p there is a point x in U such that f(x) is not in V. Since S is a first countable T_1 space there exists a sequence $\{G_n\}$ of open subsets of S such that the intersection of the elements of the sequence $\{G_n\}$ is $\{p\}$ and such that for every $m > n$, $G_m \subset G_n$. Since f is peripherally continuous at p there exists an open set D_1 containing p and contained in G_1 such that $f(F(D_1)) \subset V$. Let nl be the smallest integer such that $G_{n1} \subset D_1$. Then there exists an open set D_2 containing p and contained in G_{n1} such that $f(F(D_2)) \subset V$. Proceeding in this manner, a sequence $\{D_n\}$ of open sets can be found such that the intersection of the elements of the sequence $\{D_n\}$ is $\{p\}$, such that $m > n$ implies $D_m \subset D_n$, and such that for every n, $f(F(D_n)) \subset V$.

By the nature of V, there exists a point d_o in D_1 such that $f(d_o)$ is not in V. Now let nl be the smallest integer such that d_o is not in D_{n1}. For the same reason as before, there exists a point d_1 in D_{n1} such that $f(d_1)$ is not in V. Continuing this process inductively, a sequence $\{d_i\}$ of distinct points of S and a subsequence $\{D_{nl}\}$ of $\{D_n\}$ can be defined such that for every i, d_i is in D_{nl} and $f(d_i)$ is not in V. Since f is one-to-one, the set Q whose elements are the points $f(d_i)$ is infinite and since T is compact, then Q has a limit point q in T. Furthermore, q is not an element of V since V is open and $V \cap Q = \emptyset$. Since f maps S onto T and is one-to-one, there exists a unique point d of S such that f(d) = q. Since f(p) is in V, then $d \neq p$.

Suppose that d is in D_{nl}. Since the intersection of the elements of $\{D_n\}$ is $\{p\}$ and since $\{D_{nl}\}$ is a subsequence of $\{D_n\}$, then the intersection of the elements of $\{D_{nl}\}$ is also $\{p\}$. Hence there is a largest integer k such that d is in D_{nk}. Let r = k + 1. Then d is not in D_{nr}. Since $f(F(D_{nr})) \subset V$ and f(d) = q is not in V, it is also true that d is not in $\overline{D_{nr}}$. Hence d is in the open set $S - \overline{D_{nr}}$ and since f is an open mapping, f(d) = q is in the open set $f(S - \overline{D_{nr}})$. But if $i > r$, then $f(d_i)$ is in $f(D_{ni}) \subset f(D_{nr})$. Hence all but a finite number of the points $f(d_i)$ of Q are in the open set $f(D_{nr})$. Since f is one-to-one and D_{nr} and $S - \overline{D_{nr}}$ are disjoint open sets, then $f(D_{nr})$ and $f(S - \overline{D_{nr}})$ are also disjoint open sets. This implies that q is not a limit point of Q, a contradiction. Hence d is not in D_{nl}.

Using the same argument as above with D_{nl} in the place of D_{nr} it follows that $f(D_{nl})$ and $f(S - \overline{D_{nl}})$ are disjoint open sets containing Q and p, respectively, giving rise again to the contradiction that p is not a limit point of Q. Hence the supposition that f is not continuous at p cannot hold. Therefore f is continuous on S.

Long [6, p. 10] has shown that in general the composite of two peripherally continuous mappings is not peripherally continuous.

The following theorem shows, however, that if f is peripherally continuous and g is continuous then gf is peripherally continuous.

Theorem 3.9. Let S, R and T be topological spaces. If f is a peripherally continuous mapping of S into R and g is a continuous mapping of R into T then the composite gf is a peripherally continuous mapping of S into T.

Proof. Let x be a point of S and let U and V be open sets containing x and $g(f(x))$ respectively. Since g is continuous at $f(x)$ there exists an open set W containing $f(x)$ such that $g(W) \subset V$. Since f is peripherally continuous at x, there exists an open subset D of U containing x such that $f(F(D)) \subset W$. Since $g(f(F(D))) = (gf)(F(D)) \subset V$ then gf is peripherally continuous at x.

The following example shows that Theorem 3.9 does not remain true if the properties of f and g are interchanged and, indeed, that even if f is assumed to be both open and continuous, it does not follow that gf is peripherally continuous.

Example 3.1. Let S be the set of pairs of real numbers (x,y) such that $0 \leq x \leq 1$ and $-1 \leq y \leq 1$. Let R be the set of real numbers x such that $0 \leq x \leq 1$ and let T be the set of real numbers y such that $-1 \leq y \leq 1$. For each (x,y) in S let $f(x,y) = x$. Define g by $g(x) = \sin(1/x)$ if $x \neq 0$ and $g(0) = 0$. Then f is just the projection mapping of S onto R and is therefore both open and continuous. Since g is continuous at each $x \neq 0$ in R and is peripherally continuous at 0, then g is peripherally continuous on R.

The composite mapping gf is not peripherally continuous at any point $(0,q)$ of S. For let U be the set of points (x,y) in S such that $x^2 + (y - q)^2 < 1$ and let V be the set of points y in T such that

$-1/2 < y < 1/2$. Then U and V are open sets in S and T containing $(0,q)$ and $(gf)(0,q) = 0$, respectively. If D is any open subset of U containing $(0,q)$, then $f(F(D))$ contains an interval of the form $[0,a]$, where $a > 0$. Since $g([0,a]) = (gf)(F(D)) = [-1,1] \not\subset V$ for any such D, it follows that gf is not peripherally continuous at $(0,q)$.

CHAPTER IV

PROPERTIES OF REAL-VALUED CONNECTIVITY MAPPINGS

This chapter will be primarily concerned with the sets of points of discontinuity of connectivity mappings defined on a suitably restricted family of spaces and having values in the interval $[0,1]$. Long [6] has observed that the set of discontinuities of a connectivity mapping need not be closed and has posed the question as to whether the set of discontinuities of a connectivity mapping is a set of the first category. One of the principal results of this paper is that, for certain spaces S, there exist connectivity mappings f on S into the interval $[0,1]$ such that f is discontinuous at each point of S.

The terms defined below will be used frequently in this chapter.

Definition 4.1. If S is a topological space and A and B are subsets of S, then A is dense in B if $A \subset B \subset \overline{A}$ [3, p. 106].

Definition 4.2. If S is a topological space and A is a subset of S, then A is said to be nowhere dense if the interior of A is empty [5, p. 145].

Definition 4.3. Let S be a metric space with metric d and let A be a subset of S. The diameter of A, written $\delta(A)$, is defined by the following equations:

$\delta(A) = 0$, if A is empty,

$\delta(A) = \inf \{d(x,y) : x,y \text{ in } A\}$, if this set of numbers is bounded,

$\delta(A) = \infty$, if $\{d(x,y) : x,y \text{ in } A\}$ is not bounded [3, p. 90].

Definition 4.4. Let S be a topological space, T a metric space, and f a mapping of S into T. The oscillation of f at a point p of S, written $Q_f(p)$, is defined to be inf $\{\delta(f(U)) : U \text{ is open, } p \text{ is in } U\}$ [3, p. 126].

Note that if the space T in the above definition is a subspace of the space of real numbers, then the oscillation of f at p may be written $Q_f(p) = \inf \{\sup f(U) - \inf f(U) : U \text{ is open, } p \text{ is in } U\}$.

Definition 4.5. Let S be a topological space and f a real-valued mapping defined on S. Then f is said to be upper semi-continuous at a point p in S if, for every $\epsilon > 0$, there exists an open subset U of S containing p such that $f(x) < f(p) + \epsilon$, for every x in U. The mapping f is said to be lower semi-continuous at a point p in S if, for every $\epsilon > 0$, there exists an open subset U of S containing p such that $f(x) > f(p) - \epsilon$, for every x in U [1, p. 91].

Definition 4.6. A subset A of a topological space S is said to be a set of the first category in S if A is the union of a countable collection of nowhere dense subsets of S.

The first result concerning the discontinuities of connectivity mappings is a corollary of the following theorem.

Theorem 4.1. Let f be a connected mapping of the locally connected topological space S into the interval $I = [0,1]$. Then either the set of points of discontinuity of f is of the first category or the graph of f is not nowhere dense in $S \times I$.

Proof. For each open set U in S, define $u^*(U) = \sup f(U)$ and $v^*(U) = \inf f(U)$ and, for each x in S, let $u(x) = \inf \{u^*(U) : x \text{ in } U, U \text{ open}\}$ and let $v(x) = \sup \{v^*(U) : x \text{ in } U, U \text{ open}\}$. Let ϵ be a positive number and let x be a point of S. By definition of u^* and u, there exists an open set U containing x such that $u^*(U) - u(x) < \epsilon$. Let y be any point in U. Again, by definition of u^* and u, $u(y) \leq u^*(U)$. Hence, given $\epsilon > 0$, there exists an open set U containing x such that for every y in U, $u(y) - u(x) < \epsilon$. Therefore u is upper semi-continuous at each point x in S. By a similar argument, v is lower semi-continuous at each point of S.

Let x be any point of S. Then $u(x) - v(x) = \inf \{u^*(U) : x \text{ in } U, U \text{ open}\} - \sup \{v^*(V) : x \text{ in } V, V \text{ open}\} = \inf \{\sup f(U) : x \text{ in } U, U \text{ open}\} - \sup \{\inf f(V) : x \text{ in } V, V \text{ open}\} = \inf \{\sup f(U) : x \text{ in } U, U \text{ open}\} + \inf \{-\inf f(V) : x \text{ in } V, V \text{ open}\} = \inf \{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U \text{ open}, V \text{ open}\}$. Now, since $\{\sup f(W) - \inf f(W) : x \text{ in } W, W \text{ open}\}$ is a subset of $\{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U \text{ open}, V \text{ open}\}$, then $\inf \{\sup f(W) - \inf f(W) : x \text{ in } W, W \text{ open}\} \geq \{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U \text{ open}, V \text{ open}\}$. Let p be any number in the set $\{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U \text{ open}, V \text{ open}\}$. Then there are open sets U_p and V_p containing x such that $p = \sup f(U_p) - \inf f(V_p)$. Since $\sup f(U_p \cap V_p) \leq \sup f(U_p)$ and since $\inf f(V_p) \leq \inf f(U_p \cap V_p)$, then $\sup f(U_p \cap V_p) - \inf f(U_p \cap V_p) \leq \sup f(U_p) - \inf f(V_p) = p$. That is, given p in $\{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U \text{ open}, V \text{ open}\}$, there is a number q in $\{\sup f(W) - \inf f(W) : x \text{ in } W, W \text{ open}\}$ such that $q \leq p$. This implies that $\inf \{\sup f(W) - \inf f(W) : x \text{ in } W, W \text{ open}\} \leq \inf \{\sup f(U) - \inf f(V) : x \text{ in } U \cap V, U, V \text{ open}\}$, and therefore, by the above, that these two infimums are equal. Consequently $Q_f(x) = u(x) - v(x)$ for every x in S.

For each positive number ϵ and for each real-valued mapping g defined on S let $S(g,\epsilon)$ denote the set of points x in S such that

$Q_g(x) \geq \epsilon$. Then, for every $\epsilon > 0$, $S(u,\epsilon)$ is nowhere dense. For suppose that $\epsilon > 0$, that U is an open subset of S, and that $S(u,\epsilon)$ is dense in U. Since $S(u,\epsilon)$ is closed, then $U \subset S(u,\epsilon)$. Let x_1 be a point of U. Since u is upper semi-continuous at x_1 there exists an open set $U_1 \subset U$ containing x_1 such that $u(x) - u(x_1) < \epsilon/3$ for every x in U_1. Since $Q_u(x) \geq \epsilon$ for every x in U_1, then there exists a point x_2 in U_1 such that $u(x_2) < u(x_1) - \epsilon/3$. Again, since u is upper semi-continuous, there exists an open set $U_2 \subset U_1$ containing x_2 such that $u(x) - u(x_2) < \epsilon/3$, for every x in U_2. But, since $Q_u(x) \geq \epsilon$ for every x in U_2, then there exists a point x_3 in U_2 such that $u(x_3) < u(x_2) - \epsilon/3$. Proceeding in this manner, a sequence $\{x_n\}$ of points of U can be found such that for each n, $u(x_{n+1}) < u(x_n) - \epsilon/3$. But this implies that $u(x_{n+1}) < u(x_1) - n\epsilon/3$ and, therefore, that u is not bounded below on S. However, for every open subset U of S, $u^*(U) = \sup f(U) \geq 0$ implies that $u(x) = \inf \{u^*(U)\colon x \text{ in } U,\ U \text{ open}\} \geq 0$ for every x in S. It follows that, for every $\epsilon > 0$, $S(u,\epsilon)$ is nowhere dense. By a similar argument, for every $\epsilon > 0$, $S(v,\epsilon)$ is nowhere dense.

Suppose now that the set of discontinuities of f is not of the first category. Then there exists an $\epsilon > 0$ such that $S(f,\epsilon)$ is not nowhere dense. Let D_o be an open set such that $D_o \subset \overline{S(f,\epsilon)} = S(f,\epsilon)$. Since $S(u,\epsilon/3)$ is nowhere dense, there exists an open set $D_1 \subset D_o$ such that $D_1 \cap S(u,\epsilon/3) = \emptyset$. Since $S(v,\epsilon/3)$ is nowhere dense, there exists an open set $D_2 \subset D_1$ such that $D_2 \cap S(v,\epsilon/3) = \emptyset$. Then $D_2 \subset S(f,\epsilon)$ and $D_2 \cap [S(u,\epsilon/3) \cup S(v,\epsilon/3)] = \emptyset$. Let p be any point in D_2. Then, since $Q_u(p) < \epsilon/3$, there exists an open set D^1 containing p such that $|u(x) - u(p)| < \epsilon/3$ for every x in D^1. Similarly, there exists an open set D^2 containing p such that $|v(x) - v(p)| < \epsilon/3$ for every x

in D^2. Let $D_p = [D^1 \cap D^2] \cap D_2$. Then, for every x in D_p, both $u(x) > u(p) - \epsilon/3$ and $v(x) < v(p) + \epsilon/3$. Since $D_p \subset S(f,\epsilon)$ and since $Q_f(p) = u(p) - v(p)$, then $(u(p) - \epsilon/3) - (v(p) + \epsilon/3) = u(p) - v(p) - 2\epsilon/3 = Q_f(p) - 2\epsilon/3 \geq \epsilon/3$. Let J be the interval $(v(p) + \epsilon/3, u(p) - \epsilon/3)$. Then $J \subset [0,1] = I$ and $D_p \times J$ is open in $S \times I$. If x is in S and $u(x) > v(x)$, let J_x denote the open interval $(v(x), u(x))$. If x is in S and $u(x) = v(x)$, let $J_x = \emptyset$. Then $D_p \times J \subset \cup\{\{x\} \times J_x : x \text{ is in } S\}$ since, for every x in D_p, both $u(x) > u(p) - \epsilon/3$ and $v(x) < v(p) + \epsilon/3$, as concluded above. It will be shown that $\cup\{\{x\} \times J_x : x \text{ is in } S\}$, and therefore $D_p \times J$, is in the closure of the graph of f. In what follows the graph of f will be denoted by G(f).

Let x belong to S. Suppose $Q_f(x) = 0$. Then $J_x = \emptyset$ and therefore $\{x\} \times J_x = \emptyset \subset \overline{G(f)}$. Suppose $Q_f(x) = u(x) - v(x) > 0$ and assume that $\{x\} \times J_x \not\subset \overline{G(f)}$. Then there exists a point y in I such that (x,y) is in $\{x\} \times J_x$ but such that (x,y) is not in $\overline{G(f)}$. Let U be an open subset of $S \times I$ containing (x,y) such that $U \cap \overline{G(f)} = \emptyset$. Then there exists an open set H containing x and an open interval K containing y such that $H \times K \subset U$. Since S is locally connected it may be assumed that H is connected and, by definition of J_x, it may be assumed that K contains neither u(x) nor v(x). Note that this implies that $v(x) < y < u(x)$ for every y in K.

The point (x,u(x)) is a limit point of G(f). For let R be any open set containing x and let $(u(x) - h, u(x) + h)$, where $h > 0$, be any neighborhood of u(x). By the definition of u, there exists an open set R_h containing x such that $u(x) \leq \sup f(R_h) < u(x) + h$. But, since $\sup f(R \cap R_h) \leq \sup f(R_h)$, then $u(x) \leq \sup f(R \cap R_h)$. Hence there exists an x_h in $R \cap R_h$ such that $u(x) \leq f(x_h) \leq \sup f(R \cap R_h) \leq$

$\sup f(R_h) < u(x) + h$. That is, given any open set R containing x and any neighborhood $L = (u(x) - h, u(x) + h)$ containing $u(x)$, there exists a point $(x_h, f(x_h))$ belonging to $R \times L$. This completes the proof that $(x, u(x))$ is a limit point of $G(f)$. By a similar argument it follows that $(x, v(x))$ is a limit point of $G(f)$.

Let L_1 be an interval containing $u(x)$ such that $L_1 \cap K = \emptyset$ and let L_2 be an interval containing $v(x)$ such that $L_2 \cap K = \emptyset$. This is possible by the way the interval K was chosen. Note that if y_1 is in L_1, y is in K, and y_2 is in L_2 then $y_1 > y > y_2$. Then $H \times L_1$ and $H \times L_2$ are open, the point $(x, u(x))$ is in $H \times L_1$, and the point $(x, v(x))$ is in $H \times L_2$. Therefore, since $(x, v(x))$ and $(x, u(x))$ are limit points of $G(f)$, there exist points x_1 and x_2 belonging to H such that $(x_1, f(x_1))$ belongs to $H \times L_1$ and $(x_2, f(x_2))$ belongs to $H \times L_2$. Hence $f(x_1)$ is in L_1 and $f(x_2)$ is in L_2 and, consequently, $f(x_1)$ and $f(x_2)$ are separated in $[0,1]$ by K. Now, by the way H and K were chosen, $[H \times K] \cap \overline{G(f)} = \emptyset$. Hence, $f(H) \cap K = \emptyset$. But this implies that $f(x_1)$ and $f(x_2)$ are separated in $f(H)$ by K, which contradicts the connectedness of H and the fact that f is a connected mapping. Thus, the assumption that $\{x\} \times J_1 \not\subset \overline{G(f)}$ is false and therefore $\cup\{\{x\} \times J_x : x \text{ in } S\} \subset \overline{G(f)}$. It follows that the open set $D_p \times J \subset \overline{G(f)}$, as was to be shown. This completes the proof of the theorem.

Corollary 4.1. Let f be a connectivity mapping of the locally connected topological space S into the interval $I = [0,1]$. Then either the set of points of discontinuity of f is a set of the first category or the graph of f is not nowhere dense in $S \times I$.

Proof. Every connectivity mapping is connected [6, p. 14]. The above theorem and its corollary do not completely answer the question as to whether the set of points of discontinuity of a connectivity mapping is a set of the first category. The following theorem and its corollary demonstrates that there do exist connectivity mappings such that their sets of points of discontinuity are not sets of the first category.

Theorem 4.2. Let I be the interval $[0,1]$ and let S be a topological space such that every open subset of S contains a non-degenerate connected set and such that the product space $S \times I$ is completely normal. Suppose that there exists a collection Σ of pairwise disjoint subsets of S such that if M is any non-degenerate connected subset of S and A is any member of Σ, then $M \subset \overline{A \cap M}$. Suppose, in addition, that the cardinal number of the collection Σ is that of the topology for $S \times I$. Then there exists a connectivity mapping f of S into I such that f is discontinuous at each point of S.

Proof. Let h be a one-to-one mapping of the topology for $S \times I$ onto Σ. Consider an arbitrary point x in S. If x is in some member A of Σ, then A is the only member of Σ containing x since Σ is a pairwise disjoint collection. If x belongs to some A in Σ and $(\{x\} \times I) \cap h^{-1}(A) \neq \emptyset$, then $\{y : (x,y) \text{ is in } h^{-1}(A)\}$ has an infimum. Define a mapping f of S into I as follows:

$f(x) = 0$, if x is in no member A of Σ,

$f(x) = 0$, if x is in some A in Σ and $(\{x\} \times I) \cap h^{-1}(A) = \emptyset$,

$f(x) = \inf \{y : (x,y) \text{ is in } h^{-1}(A)\}$, if x is in A in Σ and $(\{x\} \times I) \cap h^{-1}(A) \neq \emptyset$.

It will be shown that f is a mapping having the required properties.

Suppose that f is not a connectivity mapping. Let g be the graph mapping of S into $S \times I$ induced by f. Then there exists a connected subset M of S such that $g(M) = H \cup K$, where H and K are separated subsets of $S \times I$. M is necessarily non-degenerate. Since $S \times I$ is completely normal there exist open subsets U and V of $S \times I$ containing H and K, respectively, such that $U \cap V = \emptyset$. Let A_U and A_V denote the images of U and V in Σ under the mapping h. By hypothesis, $A_V \cap M$ and $A_U \cap M$ are each dense in M.

Let x be a point in $A_U \cap M$ and suppose that $f(x) > 0$. Then, by definition of f, $(\{x\} \times I) \cap U \neq \emptyset$. Now the open set U cannot contain just the point $(x,1)$ of $(\{x\} \times I) \cap U$ since $(x,1)$ is a limit point of $[(\{x\} \times I) \cap U] - \{(x,1)\}$. Hence there is a number y, $0 \leq y < 1$, such that (x,y) is in U. Consequently, by the definition of f, $0 < f(x) \leq y < 1$.

Let G be any open subset of $S \times I$ containing $(x,f(x))$ and let W and J be open subsets of S and I containing x and $f(x)$, respectively, such that $W \times J \subset G$. Again, by definition of f, J contains a point y_1 such that $f(x) < y_1$ and such that (x,y_1) is in U. J also contains a point y_2 such that $y_2 < f(x)$. The point (x,y_2) cannot belong to U. Hence $W \times J$, and therefore G, contains a point (x,y_1) in U and a point (x,y_2) not in U. Since G was an arbitrary open set containing $(x,f(x))$, it follows that $(x,f(x))$ is in the boundary of U. But this contradicts the fact that $(x,f(x)) = g(x)$ is in $g(M) \subset U \cup V$ and that $U \cap V = \emptyset$, since $g(M)$ is connected and U and V are open. Hence for every x in $A_U \cap M$, $f(x) = 0$. Similarly, for every x in $A_V \cap M$, $f(x) = 0$.

Let U_1 and V_1 denote the projections of U and V onto S. Then U_1 and V_1 are open in S and $U_1 \cup V_1$ contains M since $U \cup V$ contains $g(M)$. Furthermore, $U_1 \cap V_1$ contains a point of M. Otherwise, the relation

$\emptyset = (U_1 \cap V_1) \cap M = (U_1 \cap M) \cap (V_1 \cap M)$ implies that M is the union of two disjoint sets each open in M which in turn implies that M is not connected. Since both $A_U \cap M$ and $A_V \cap M$ are dense in M, then $U_1 \cap V_1$ contains both points of $A_U \cap M$ and points of $A_V \cap M$. Let x be any point in $(U_1 \cap V_1) \cap (A_U \cap M)$. By the above, $f(x) = 0$, and since x is in the projection U_1 of U onto S, then $(\{x\} \times I) \cap U \neq \emptyset$. Let G be any open subset of $S \times I$ containing $(x,0) = (x,f(x))$. Then there is an open set $J \subset S$ containing x and an open set $J \subset I$ containing f(x) such that $W \times J \subset G$. Since $f(x) = 0$, it follows from the definition of f that J contains a point y such that (x,y) is in U. Hence (x,0) is a limit point of U. But then (x,0) must belong to U since $g(x) = (x,f(x)) = (x,0)$ is in $U \cup V$ and since $U \cap V = \emptyset$. Similarly, if x is in $(U_1 \cap V_1) \cap (A_V \cap M)$, then $g(x) = (x,0)$ is in V.

Now let x be any point of $(U_1 \cap V_1) \cap (A_V \cap M)$. By the above, $g(x) = (x,f(x)) = (x,0)$ is in V. Hence there is an open set G containing g(x) such that $G \subset V$ and such that G is of the form $W \times J$, where W is an open subset of S containing x and J is an open subset of I containing 0. The open set W can be chosen such that $W \subset U_1 \cap V_1$. Since $A_U \cap M$ is dense in M, then W contains a point z of $A_U \cap M$. But z in $(U_1 \cap V_1) \cap (A_U \cap M)$ implies that $g(z) = (z,0)$ is in U. Hence $U \cap V \neq \emptyset$, a contradiction. It follows that the supposition that f is not a connectivity mapping can not hold.

It remains to be shown that f is discontinuous at every point of S. Let x_o be any point of S, y_o a point of I such that $0 \leq y_o < 1$, and W any open subset of S containing x_o. Let b be a point in I such that $y_o < b < 1$, let J be the open interval (y_o,b), and let $U = W \times J$. By hypothesis, the open set W contains a non-degenerate connected subset M of S and the member $h(U) = A_U$ of Σ is such that $A_U \cap M$ is dense in M. Let z be a point of $A_U \cap M$. Then $f(z) = \inf \{y : (x,y) \text{ is in } U\} = \inf \{y : y_o < y < b\} = y_o$. Hence f assumes every value y_o between 0 and 1 in every open set W containing x_o. Therefore f is not continuous at x_o. Since x_o was chosen arbitrarily, then f is discontinuous at every point of S.

Corollary 4.2. There exists a connectivity mapping f of the set R of real numbers into the interval [0,1] such that f is discontinuous at each point of R.

Proof. Let Q denote the rational number field and let R(Q) denote the vector space of the reals over the rationals. Then R(Q) has a basis B; that is, there exists a subset B of R such that for every $x \neq 0$ in R there corresponds a unique finite set $\{q_1, q_2, \ldots, q_n\}$ of non-zero rationals and a unique finite subset $\{b_1, b_2, \ldots, b_n\}$ of B such that $x = \sum_{k=1}^{n} q_k b_k$ [1, p. 150]. The set B is uncountable. To show this, assume B is countable and arrange the elements of B into a sequence $\{b_1, b_2, \ldots, b_n, \ldots\}$. For each n, let $L_n = \{\sum_{k=1}^{n} q_k b_k : q_k \text{ is in } Q\}$. Then each L_n is countable and, therefore, $\cup\{L_n : n = 1,2,3,\ldots\} = R$ is countable, a contradiction. Hence the cardinal number of B is c, the cardinal number of the real numbers. For each b in B let $A_b = \{qb : q \text{ is in } Q,\ q \neq 0\}$ and let $\Sigma = \{A_b : b \text{ in } B\}$. Then the cardinal number of Σ is c and each A_b in Σ is dense in R. Hence, for each member A_b of Σ, $A_b \cap M$ is dense in every non-degenerate connected subset M of R. For each pair b, b* of distinct members of B, $A_b \cap A_{b*} = \emptyset$. Otherwise, there would exist non-zero rational numbers q_1 and q_2 such that $q_1 b = q_2 b^*$, contradicting the fact that B is a basis for R(Q).

Since both R and $I = [0,1]$, with the usual topologies, are completely normal then $R \times I$ is also completely normal [3, p.110]. The space $R \times I$ is also second countable, that is, there is a countable basis B^* for its topology. Thus, if 2^{B^*} denotes the collection of subcollections of B^*, then the cardinal number of 2^{B^*} is c [1, p. 25]. To each open subset U of $R \times I$ there corresponds a unique subcollection of B^*, namely the elements of B^* that are contained U. Hence the open subsets of $R \times I$ can be put into one-to-one correspondence with a subset of 2^{B^*}. Since there are at least c open subsets of $R \times I$, it follows that the topology for $R \times I$ has cardinal number c. Clearly, every open subset of R contains a nondegenerate connected subset. Hence the space R satisfies the conditions stipulated in the hypothesis of Theorem 4.2 and, consequently, the existence of the mapping f is proven.

Theorems 4.1 and 4.2 give no information concerning the set of points of discontinuity of connectivity mappings of the closed n-cell I_n into itself if $n \geq 2$. In [4, p. 754] Hamilton gave an example of a connectivity mapping of the closed n-cell I_n into itself which had a single point of discontinuity and in [2] Hagan posed the question as to whether a connectivity mapping of I_n into itself can be discontinuous on a dense subset of I_n. The following example shows that there exist peripherally continuous mappings of I_n into itself, and therefore, in view of [2 p. 17], connectivity mappings of I_n into itself with dense sets of points of discontinuity.

Example 4.1. This is an example of a peripherally continuous mapping of an N-cell I into itself that is not continuous at each point of a set X which is dense in I.

If x and y are two points in Euclidean N-space E_N and a and b are real numbers, the expressions $ax + by$, $|x - y|$ and $|x|$ will denote the point $(ax_1 + by_1, ax_2 + by_2, \ldots, ax_N + by_N)$, the distance between x and y and the distance from x to the origin $0 = (0,0,\ldots,0)$, respectively.

Let I be the closed spherical N-cell with radius 1 and center 0 and let $\{x_n\}$ be a sequence of distinct interior points of I such that, for all n, $x_n \neq 0$ and such that the point set union X of the points of the sequence $\{x_n\}$ is dense in I. Let $\{D_n\}$ be a sequence of closed spherical N-cells such that, for each n, D_n is interior to I, D_n has as center the point x_n of the set X and, if $k < n$, x_k is not in D_n. Let r_n denote the radius of D_n. For each integer n, define the mapping g_n by:

$$g_n(x) = x, \text{ if } x \text{ is not in } D_n - \{x_n\},$$

$$g_n(x) = x_n + (1/2)r_n\left\{1 + \sin\frac{r_n\pi}{2|x - x_n|}\right\}\frac{x - x_n}{|x - x_n|}, \text{ if } x \text{ is in } D_n - \{x_n\}.$$

Then, if x is in $D_n - \{x_n\}$, $|g_n(x) - x_n| \leq r_n$, and if $x = x_n$, then $|g_n(x) - x_n| = 0$. Hence $g_n(D_n) \subset D_n$. If x is in the boundary $F(D_n)$ of D_n then $|x - x_n| = r_n$ and, consequently,

$$g_n(x) = x_n + (1/2)r_n\left\{1 + \sin\frac{\pi}{2}\right\}\frac{x - x_n}{r_n} = x_n + x - x_n = x.$$

Examination of the relations defining g_n shows g_n to be continuous at every point of D_n except possibly at x_n. Since g_n is the identity mapping on $I - D_n$ and also on the boundary $F(D_n)$ of D_n, then g_n is continuous at all points of I except possibly at x_n. Actually g_n is not continuous at x_n. For consider the sequence $\{y_k\}$, where,

$$y_k = x_n + \frac{r_n}{(4k + 1)|x_n|} x_n.$$

Since,

$$|y_k - x_n| = \left| \frac{r_n}{(4k + 1)|x_n|} x_n \right| = \frac{r_n}{(4k + 1)|x_n|} |x_n| = \frac{r_n}{4k + 1} < r_n,$$

then y_k is in D_n and, furthermore, $\lim_{k\to\infty} y_k = x_n$.

Now, since,

$$|g_n(y_k) - x_n| = |(1/2)r_n\{1 + \sin \frac{(4k + 1)\pi}{2}\} \frac{y_k - x_n}{|y_k - x_n|}| = r_n,$$

then $g_n(y_k)$ is on the boundary $F(D_n)$ of D_n, for each k. Therefore, since, $\lim_{k\to\infty} y_k = x_n$, the center of D_n, and since $g_n(x_n) = x_n$, it follows that g_n is not continuous at x_n.

However, g_n is peripherally continuous at x_n. For consider the sequence $\{S_k\}$ of spheres such that x_n is the center of each S_k and, for each k, $r_n/(4k + 3)$ is the radius of S_k. Then, since $r_n/(4k + 3) < r_n$, x in S_k implies that x is in D_n and, therefore, that

$$|g_n(x) - x_n| = |(1/2)r_n\{1 + \sin \frac{(4k + 3)\pi}{2}\} \frac{x - x_k}{x - x_n}|$$

$$= (1/2)r_n \{1 + \sin \frac{3\pi}{2}\} = 0.$$

Thus, $g_n(S_k)$, the image of the boundary of the open n-cell with center x_n and radius $r_n/(4k + 3)$, is the singleton set $\{x_n\}$, which is a subset of any open set containing $g_n(x_n) = x_n$. Since every open set containing x_n contains one of the spheres S_k, it follows that g_n is peripherally continuous at x_n. This, together with the fact that g_n is continuous at all other points of I, shows that g_n is peripherally continuous on I.

Define the sequence $\{f_n\}$ of mappings of I into E_N by :

$$f_n(x) = \sum_{k=1}^{n} 2^{-k} g_k(x).$$

Then, for every x in I,

$$|f_n(x)| = |\sum_{k=1}^{n} 2^{-k} g_k(x)| \leq \sum_{k=1}^{n} 2^{-k} g_k(x) \leq \sum_{k=1}^{n} 2^{-k} < 1.$$

Hence $f_n(I) \subset I$.

Now, the relation defining f_n shows that f_n is continuous at every point x in I, with the exceptions $x_1, x_2, x_3, \ldots, x_n$, since each of the mappings g_k, $1 \leq k \leq n$, is continuous at all points of I except these points. In addition, f_n is peripherally continuous at each of the points $x_1, x_2, x_3, \ldots, x_n$. For suppose that $1 \leq y \leq n$ and let $\{S_i\}$ be a sequence of spheres, each with center x_j, such that the sphere S_i has radius $r_j/(4i + 3)$. Thus, for each i, $r_j/(4i + 3) < r_j$ implies that, if x is in S_i, then x is in D_j. Therefore, using the same argument as that above, with g_j in the place of g_n, it follows that if x is in S_i, then $g_j(x) = x_j$. Now, each of the other mappings g_k, where $1 \leq k \leq n$ and $k \neq j$, is continuous at x_j, as was noted previously. Hence the mapping h, defined on I by

$$h(x) = \sum_{k=1}^{j-1} 2^{-k} g_k(x) \quad + \quad \sum_{k=j+1}^{n} 2^{-k} g_k(x),$$

is continuous at x_j. Also, $f_n(x) = h(x) + g_j(x)$, for each x in I.

Let U and V be open sets containing x_j and $f_n(x_j)$ respectively. Since the vector operation of addition is continuous on $E_N \times E_N$ to E_N and since $f_n(x_j) = h(x_j) + g_j(x_j)$, then there exists a pair H, G of open sets, containing $h(x_j)$ and $g_j(x_j)$ respectively, such that if x is in H and y is in G, then x + y is in V. Since h is continuous at x_j, then there exists an open subset U_1 of U such that $h(U_1) \subset H$. By the definition of the S_i, there exists an integer i_o such that $S_{i_o} \subset U_1$. Let W be the open N-cell with boundary S_{i_o}. Then x in $F(W) = S_{i_o}$ implies that h(x) is in H and that $g_j(x) = x_j$ is in G. Hence, from the continuity of vector addition, if x is in $S_{i_o} = F(W)$, then $h(x) + g_j(x) = f_n(x) \in V$. This completes the proof that f_n is peripherally continuous at x_j, $1 \le j \le n$. This, combined with the fact that f_n is continuous at all other points of I, shows f_n to be peripherally continuous on I.

For each x in I, the sequence $\{f_n(x)\}$ is a Cauchy sequence since, if $\epsilon > 0$, $m > \log_2 \epsilon^{-1}$ and, $m < n$, then

$$\begin{aligned}
|f_n(x) - f_m(x)| &= \left| \sum_{k=1}^{n} 2^{-k} g_k(x) - \sum_{k=1}^{m} 2^{-k} g_k(x) \right| \\
&= \left| \sum_{k=m+1}^{n} 2^{-k} g_k(x) \right| \\
&\le \sum_{k=m+1}^{n} 2^{-k} |g_k(x)| \\
&\le \sum_{k=m+1}^{n} 2^{-k} \\
&< 2^{-m} < \epsilon.
\end{aligned}$$

Thus, a mapping f can be defined on I by

$$f(x) = \underset{n\to\infty}{\text{limit}}\ f_n(x) = \underset{n\to\infty}{\text{limit}} \sum_{k=1}^{n} 2^{-k} g_k(x) = \sum_{k=1}^{\infty} 2^{-k} g_k(x).$$

Let x belong to I. Then,

$$\begin{aligned} |f(x)| &= \left| \sum_{k=1}^{\infty} 2^{-k} g_k(x) \right| \\ &\leq \sum_{k=1}^{\infty} 2^{-1} |g_k(x)| \\ &\leq \sum_{k=1}^{\infty} 2^{-k} = 1. \end{aligned}$$

Hence $f(x)$ is in I, so $f(I) \subset I$.

Now, let x be any point of I and let ϵ be any positive number. Let M be an integer larger than $\log_2 \epsilon^{-1}$. Then, if $n > M$,

$$\begin{aligned} |f(x) - f_n(x)| &= \left| \sum_{k=1}^{\infty} 2^{-k} g_k(x) - \sum_{k=1}^{n} 2^{-k} g_k(x) \right| \\ &= \left| \sum_{k=n+1}^{\infty} 2^{-k} g_k(x) \right| \\ &\leq \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n} < \epsilon. \end{aligned}$$

Therefore, $\{f_n\}$ converges uniformly to f on I. Hence, by a theorem of Hagan [2, p. 13], f is peripherally continuous on I. It remains to be shown that each of the points x_n, $n = 1,2,3,\ldots$, is a point of discontinuity of f.

Let x_n be one of these points. Then, for every $m > n$, the mapping h_m, defined by

$$h_m(x) = \sum_{k=n+1}^{m} 2^{-k} g_k(x),$$

is continuous at x_n since each of the mappings g_k, where $n + 1 \leq k \leq m$, is continuous at x_n. Hence, using an argument similar to that above, with h_m in the place of f_n, it follows that $\{h_m(x)\}$ is a Cauchy sequence in E_N and therefore $\underset{m \to \infty}{\text{limit}}\ h_m(x)$ exists. Let

$$h_n(x) = \underset{m \to \infty}{\text{limit}}\ h_m(x) = \underset{m \to \infty}{\text{limit}} \sum_{k=n+1}^{m} 2^{-k} g_k(x) = \sum_{k=n+1}^{\infty} 2^{-k} g_k(x).$$

Again, using an argument similar to the one above, with h_m in place of f_n and h_n in place of f, it follows that the sequence $\{h_m\}$ converges uniformly to h_n on I. Therefore, since x_n is a point of continuity of h_m for all $m = n + 1, n + 2, n + 3, \ldots$, then x_n is a point of continuity for h_n.

Recall that f_{n-1} is continuous at x_n. By noting that for any x in I, $f(x) = f_{n-1}(x) + 2^{-n}g_n(x) + h_n(x)$, it can be seen that x_n cannot be a point of continuity of f. Otherwise g_n would be continuous at x_n, contrary to the fact proven earlier that x_n is the only point at which g_n is not continuous.

Since it has been shown that, for each integer n, f_n is continuous at every point of I except at each of the points $x_1, x_2, \ldots, x_n$, and since $\{f_n\}$ converges uniformly to f on I, then f is also continuous at any $x \neq x_n$. Thus f is a peripherally continuous mapping of I into itself whose set of points of discontinuity is the set X which is dense in I.

The following is an example of a connected mapping f of the real number space R into itself such that the graph of f is a totally disconnected subset of R × R; that is, such that the graph of f contains no nondegenerate connected subsets. It gives, therefore, some measure of the difference between connected mappings and connectivity mappings. The construction of f is similar to that in [1, p. 150] but differs in that it is specifically done in such a way to make the graph of f totally disconnected.

<u>Example 4.2.</u> This is an example of a connected mapping f of the real number space R into itself such that the graph of f is a totally disconnected subset of $R \times R$.

Let R(Q) denote the vector space of the real numbers over the rational number field Q. Let b_o be a positive rational number and let B denote an extension of $\{b_o\}$ to a basis for R(Q). Then, as was shown in the proof of Corollary 4.2, B has cardinal number c, the cardinal number of the real numbers. Let R^+ denote the set of positive reals and let R^o denote the set of non-negative reals.

In the following construction it will be necessary to show the existence of a one-to-one mapping of $B - \{b_o\}$ onto B with the property that, for every finite subset N of $B - \{b_o\}$, $f(N) \neq N$. The existence of such a mapping will be proven if the existence of a one-to-one mapping f^* of R^+ onto R^o can be found such that $f(N) \neq N$, for every finite subset N of R^+. For suppose that such a mapping f^* can be found and let F be a one-to-one mapping of B onto R^o with the property that $F(b_o) = 0$. Let F also denote the restriction of F to $B - \{b_o\}$. This is not likely to cause any confusion in what follows. Then F (actually the restriction of F to $B - \{b_o\}$) is a one-to-one mapping from $B - \{b_o\}$ onto R^+, f^* is, by assumption, a one-to-one mapping of R^+ onto R^o, and F^{-1} is a one-to-one mapping of R^o onto B. Therefore, the composite, $F^{-1}f^*F$, of these mappings is a one-to-one mapping of $B - \{b_o\}$ onto B. Let f denote this composite mapping.

Assume that there exists a finite subset N of $B - \{b_o\}$ such that $f(N) = N$. Then, from the definition of f, $f(N) = F^{-1}[f^*(F(N))]$. Hence, $F(f(N)) = F\{F^{-1}[f^*(F(N))]\}$ implies that $F(f(N)) = f^*(F(N))$. Finally, substituting N for f(N), $F(N) = f^*(F(N))$. But this contradicts the property of f^* that $f^*(F(N)) \neq F(N)$, for any finite subset $F(N)$ of $B - \{b_o\}$. Hence, if a mapping f^* as described above exists then the required mapping f also exists.

To prove the existence of such a mapping f^* consider the mapping f^* defined on R^+ by:

$$f^*(x) = x^2 \text{ , if } n \text{ is not an integer,}$$

$$f^*(n) = (n-1)^2 \text{, if } n \text{ is a positive integer}$$

Then the graph of f^* in $R \times R$ can be obtained by deleting the points (n,n^2) from the graph of the function g, defined by $g(x) = x^2$, if $x > 0$, and by adding the points $(n,(n-1)^2)$ to the set that remains. Examination of the resulting set shows that, for every $y \geq 0$, there is a unique $x > 0$ such that $f^*(x) = y$. Hence f^* is a one-to-one mapping of R^+ onto R^o.

To show that f^* has the property that $f^*(N) \neq N$ for every finite subset N of R^+, let N be a finite subset of R^+ and let x_N denote $\min\{x : x \in N\}$. If $x_N \leq 1$, then $f^*(x_N) < x_N$ implies $f^*(N) \neq N$. If $x_N > 1$ and 2 is not in N, then $x_N < f^*(x)$, for every x in N, implies $f^*(N) \neq N$. If $x_n > 1$ and 2 is in N then $f^*(2) = 1 \notin N$ implies that $f^*(N) \neq N$. Hence $f^*(N) \neq N$ for any finite subset N of R^+ and, therefore, the existence of the required mapping f of $B - \{b_o\}$ onto B is established.

Extend the domain of f to the real numbers as follows: let $f(b_o) = 0$, and if x is a number not in B, let $f(x) = \sum_{i=1}^{n} q_i f(b_i)$, where $\sum_{i=1}^{n} q_i b_i$ is the unique representation of x as a linear combination of elements of B using scalars in Q. Then, for any two reals x and y, and for any two rationals p and q, $f(px + qy) = pf(x) + qf(y)$. In particular, if p is a rationsl, then $p = qb_o$, for some rational q. Hence $f(p) = f(qb_o) = qf(b_o) = 0$.

Let (a,b) be any open interval and let y be any real number. Then f assumes the value y in (a,b). To show this, consider the unique representation $\sum_{i=1}^{n} q_i b_i$ for y and let $b^*_i = f^{-1}(b_i)$, $i = 1,2,3,\ldots,n$. Then, if $x = \sum_{i=1}^{n} q_i b^*_i$, $f(x) = \sum_{i=1}^{n} q_i f(b^*_i) = \sum_{i=1}^{n} q_i b_i = y$. Now the collection of rational multiples of b_o is dense in R since this set is exactly the set of rationals. Hence $\{x + qb_o : q \in Q\}$ is also dense in R. Choose a number q such that $x + qb_o$ is in (a,b). Then $f(x + qb_o) = f(x) + qf(b_o) = f(x) = y$. Hence, $f[(a,b)] = R$, for any open interval (a,b). This implies that f is a connected mapping.

It will be proven now that if r is rational, then the line with equation $y = x + r$ contains no points of the graph of f, unless $r = 0$, in which case only the point (0,0) of the graph of f lies on this line. Let x be rational. Then $x = qb_o$, for some rational number q, and $f(x) = qf(b_o) = 0$. Consequently, $(x,f(x))$ is not on $y = x + r$ for any rational number r, unless $x = 0$.

Suppose now that x is not rational and that $f(x) = x + r$, for some rational r. Let $q_1b_1 + q_2b_2 + \ldots + q_nb_n$ be the unique representation for x in terms of the basis B, where the q_i are non-zero rationals. If b^*_i represents $f(b_i)$, for each i, it follows that $q_1b^*_i + \ldots + q_nb_n = qb_o + q_nb_n$, where q is a rational such that $r = qb_o$.

Consider first the case where $n = 1$. Then the equation last written reduces to $q_1b^*_i = qb_o + q_1b_1 = x + r$. Since $x + r$ has a unique representation in terms of the elements of B and since $q_1 \neq 0$ it follows that $q = 0$. But then $q_1b^*_i = q_1b_1$ implies that $b^*_1 = f(b_1) = b_1$, which contradicts the fact that $f(N) \neq N$ for any finite subset N of $B - \{b_o\}$. Consequently it must be the case that $n > 1$. Then, refering again to the equation $q_1b_1^* + \ldots + q_nb_1^* = qb_o + q_1b_1 + \ldots + q_nb_n$, and to the fact that $x + r$ has a unique representation in terms of the b_i, it must be concluded that either $q = 0$ or that $b_k = b_o$ for some k, $1 \leq k \leq n$.

Suppose first that $q = 0$. Then, by the uniqueness of the representation for $x + r$, the set $N = \{b_1, b_2, \ldots, b_n\}$ is the same as $\{b_1^*, \ldots, b_n^*\}$, again contradicting the fact that $f(N) \neq N$, for every finite subset N of $B - \{b_o\}$.

It must be the case, therefore, that $b_k = b_o$, for some k, where $1 \leq k \leq n$. For convenience, assume the notation chosen in such a way that $b_n = b_o$. Then $q_1b_1^* + \ldots + q_1b_n^* = (q + q_n)b_o + q_1b_1 + \ldots + q_{n-1}b_{n-1}$. Since $b_n = b_o$ and since $b_n^* = f(b_n) = f(b_o) = 0$, then this equation reduces to $q_1b_1^* + \ldots + q_{n-1}b_{n-1}^* = (q + q_n)b_o + q_1b_1 + \ldots + q_{n-1}b_{n-1}$. Since $x + r$ cannot be represented both as a linear combination of n, and as a linear combination of $n - 1$ of the elements of B, using non-zero scalars q_i as coefficients, it follows that $q + q_n = 0$. Hence $q_1b_1^* + \ldots + q_{n-1}b_{n-1}^* = q_1b_1 + \ldots + q_{n-1}b_{n-1}$ and, furthermore; $q_i \neq 0$ for any $i = 1,2,3,\ldots,n - 1$. Therefore the set $N = \{b_1, b_2, \ldots, b_{n-1}\}$ is identical to $f(N) = \{b_1^*, b_2^*, \ldots, b_{n-1}\}$, which gives rise again to the contradiction that $f(N) = N$ for a finite subset N of $B - \{b_o\}$. Consequently, since all possibilities have been exhausted, it must be concluded that if x is not rational then $f(x) \neq x + r$, for any rational number r. This completes the argument that the only point of the graph of f that lies on a line with equation $y = x + r$, for some rational number r, is the point (0,0). It follows, by a similar argument, that $(0,0)$ is also the only point of the graph of f that lies on a line with equations $y = r - x$, where r is rational. Consequently, the graph of f is totally disconnected.

CHAPTER V

SUMMARY

This paper is primarily concerned with a certain type of non-continuous mapping, namely the F_2-mapping, and with the discontinuities of some particular F_2-mappings; the connectivity mappings.

The F_1-mappings and F_2-mappings are defined in Chapter II and the relationships of these mappings to the continuous mappings, connected mappings, and connectivity mappings are established. Necessary and sufficient conditions that a mapping f be an F_2-mapping are given in terms of the connected subsets of the domain of f. Sufficient conditions that a mapping be an F_2-mapping are given, also in terms of the connected subsets of the domain of the mapping. It is found that f is an F_2-mapping if and only if, for every connected subset M of its domain, $f(\overline{M}) \subset \overline{f(M)}$. A sufficient condition that a mapping f be an F_1-mapping is that, for every connected subset M of the domain of f, $f(\overline{M})$ is a subset of the union of the closures of the components of f(M).

The operation of composition of functions is considered in connection with these mappings and it is shown that the collection of F_1-mappings is closed under this operation but that the composite of two F_2-mappings is not necessarily an F_2-mapping. As a final result of Chapter II, the limit of a uniformly convergent sequence of F_2-mappings is an F_2-mapping.

In Chapter III attention is turned to the graph mappings and projection mappings induced by F_2-mappings. It is found that, if the graph mapping g induced by f is an F_2-mapping, then f is an F_2-mapping and if, in addition, the domain of f is hereditarily locally connected, then f being an F_2-mapping implies that g is an F_2-mapping. Without any restrictions on the spaces involved, it is shown that f is an F_2-mapping if and only if its induced projection mapping is an F_2-mapping.

Also in Chapter III, some conditions that imply the continuity of the projection mapping are given. The projection mapping h is continuous if and only if f is continuous. If f is a mapping of a space S into a connected space T and if h is peripherally continuous, then h is continuous.

One of the main results of this paper is contained in Chapter IV. If I is the interval [0,1] and S is a topological space such that $S \times I$ is completely normal and such that there exists a collection Σ of pairwise disjoint subsets of S having the property that the intersection of any member of Σ with a nondegenerate connected subset of M is dense in M and having, in addition, the property that the cardinal number of Σ is that of the topology for $S \times I$, then there exists a connectivity mapping of S into I which is continuous at no point of S. A corollary of this result is that there exists a connectivity mapping of the reals R into I which is continuous at no point of R. In response to a question of Long [6], it is shown that either the set of points of discontinuity of a connectivity mapping f, on a locally connected topological space S into the interval I, is a set of the first category or the graph of f is not nowhere dense in $S \times I$.

An example is given to show the existence of a connectivity mapping f of the closed n-cell I_n into itself such that f is discontinuous at each point of a dense subset of I_n. An example of a connected mapping on the reals R into R is provided to illustrate the difference between the notions of connected mappings and connectivity mappings.

Some questions for further study might include the following. Are the conditions stipulated in Theorem 2.5 involving the connected subsets of the domain space of a mapping f both necessary and sufficient in order that f be an F_1-mapping? Under what conditions will an F_2-mapping be a connectivity mapping? Under what conditions will an F_2-mapping be continuous? Does an F_2-mapping of the closed n-cell I_n into itself that has only a finite number of points of discontinuity leave a point of I_n fixed? Does the Pseudo-Arc have the fixed point property with respect to connectivity mappings or with respect to peripherally continuous mappings? To the writer's knowledge the question posed by Long [6] concerning the nature of the set of points of discontinuity of a peripherally continuous mapping of a closed n-cell into itself remains unanswered.

Finally, which subsets of Euclidean n-space, E_n, are images of the interval [0,1] under connectivity mappings? If M is a connected subset of E_n does there exist a connectivity mapping from [0,1] onto M?

BIBLIOGRAPHY

1. Goffman, Casper, Real Functions, New York: Rhinehart and Company Inc., 1953.

2. Hagan, Melvin, Properties of Peripherally Continuous Functions and Connectivity Maps, Doctorial Thesis, Oklahoma State University, 1964.

3. Hall, D. W., and Spencer, G. L., Elementary Topology, New York: John Wiley and Sons, 1955.

4. Hamilton, O. H., Fixed Points for Certain Non-continuous Transformations, Proc. Amer. Math. Soc., Vol. 8 (1957), pp. 750-756.

5. Kelly, John L, General Topology, New York: D. Van Nostrand Company, Inc., 1955.

6. Long, Paul E., Properties of Certain Non-continuous Transformations, Doctorial Thesis, Oklahoma State University, 1961.

7. Nash, John, Generalized Brouwer Theorem, Bull. Amer. Math. Soc., Research Problem 62-1-76.

8. Stallings, J., Fixed Point Theorems for Connectivity Maps, Fund. Math., Vol. XLVIII 3 (1959), pp. 249-263.

9. Whyburn, G. T., Analytic Topology, Amer. Math. Soc., Colloquim Publications, Vol. 28, 1963.

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